July, 1995 LBL-37524
UCB-PTH-95/26

General Classical Solutions of Nonlinear σ-Model and Pion Charge Distribution of Disoriented Chiral Condensate*

Zheng Huang† and Mahiko Suzuki‡

†*Department of Physics, University of Arizona, Tucson, AZ 85741*
‡*Theoretical Physics Group, Lawrence Berkeley Laboratory*
University of California, Berkeley, CA 94720, USA

Abstract

We obtain the general analytic solutions of the nonlinear σ-model in $3+1$ dimensions as the candidates for the disoriented chiral condensate (DCC). The nonuniformly isospin-orientated solutions are shown to be related to the uniformly oriented ones through the chiral (axial) rotations. We discuss the pion charge distribution arising from these solutions. The distribution $dP/df = 1/(2\sqrt{f})$ holds for the uniform solutions in general and the nonuniform solutions in the $1+1$ boost invariant case. For the nonuniform solution in $1+1$ without a boost-invariance and in higher dimensions, the distribution does not hold in the integrated form. However, it is applicable to the pions selected from a small segment in the momentum phase space. We suggest that the nonuniform DCC's may correspond to the mini-Centauro events.

PACS Numbers: 11.30.Rd, 11.80.La, 12.38.Mh

---

*This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under Grant PHY-90-21139.

‡ Also at Department of Physics, University of California, Berkeley, CA 94720.

1 Introduction

The soft particle production in a very high energy hadron-hadron or nucleus-nucleus collision is an interesting phenomenon. Occasionally, the collision creates a large number of low energy (small p_t) particles, mainly pion quanta, initially populating in a small interaction volume and subsequently undergoing a rapid expansion. The perturbative QCD is not applicable in describing the dynamics since it involves a large number of quanta and the interactions are highly nonlinear. One may anticipate some novel dynamical feature of the nonperturbative QCD. Although there is some evidence that the p_t distribution of these particles follows the scaling law with an effective temperature, it is not clear whether or not these low energy particles can actually thermalize so that their distribution can be described by thermodynamics. In fact, some deviation from the thermal distribution in the very small p_t region, say, $p_t < 100$ MeV, has been observed though data are poor in this region at present.

On the other hand, it has been suggested, first by Horn and Silver [1], that these low energy pions may be described by a classical theory. The number (N) of quanta involved is large, the quantum fluctuation is suppressed by $1/\sqrt{N}$. In addition, the low energy theorem on the pion-pion scatterings dictates that the quantum corrections to the scattering amplitudes are suppressed by a factor of $p^2/(4\pi f_\pi)^2$. More recently, a scenario of disoriented chiral condensate [2, 3] suggests that these low energy pions may be out of equilibrium and undergo a quench following a chiral phase transition, and their interactions should be described by the classical chiral dynamics.

In this paper, we shall determine the possible classical field evolutions that these low energy pions may follow based on the nonlinear σ-model. The advantage of the *nonlinear* σ-model over the *linear* σ model is that the constraint of the vacuum expectation value on the fields is built in and that the pion fields always describe the massless modes irrespective of the vacuum orientation in the background. The σ mass is taken to be infinity so the low energy structure of the theory is evident. We have obtained in an analytic form a class of classical solutions to the nonlinear σ-model in $3+1$ space-time dimensions as the candidates of the disoriented chiral condensate in QCD. Our general solutions have a transverse momentum distribution and need not

be subject to a boost-invariance constraint. The solution with a nonuniform isospin orientation is constructed by the chiral $SU(2) \times SU(2)$ rotation from a uniformly oriented solution. In the limit of a boost invariance and no transverse momentum, our solutions reduce to those of Blaizot and Krzywicki [4]. We study the distribution of the neutral pion fraction f for the pions that disintegrate from the disoriented vacua. We find that the distribution $\mathrm{dP/df} = 1/(2\sqrt{f})$ holds for the uniformly oriented vacua and also for the boost-invariant vacua with an infinitely large uniform spread in the transverse direction, but it does not hold for the vacuum whose isospin orientation is nonuniform in space-time. However this distribution should be correct if one selects pions from within a small region in the y-$\mathbf{k}_\perp$ plot event by event. This conclusion is reached through the analysis using the classical field theory method and also by studying the quantum pion states.

We organize the paper as follows. In Sec. 2, we start with the analysis for the boost-invariant solutions with no transverse momentum. In Sec. 3, we make the observation that all solutions with a nonuniform isospin orientation are obtained by the chiral rotations from a uniformly oriented solution whose energy is degenerate with the nonuniform ones. In Sec. 4, we give a general solution with a uniform isospin orientation, from which we can obtain the nonuniform solutions by the chiral rotations according to the prescription given in Sec. 3. The general solution has a nontrivial transverse momentum distribution and is not subject to the boost-invariance constraint. In Sec. 5, we examine the charge distribution of the pions disintegrating from these disoriented vacua. The picture of classical field theory leading to the distribution $\mathrm{dP/df} = 1/(2\sqrt{f})$ does not apply to the nonuniformly oriented vacua except in the boost invariant limit with zero transverse momentum. For a general solution which has rapidity and transverse momentum dependence, the distribution holds only within each small segment in the y-$\mathbf{k}_\perp$ plot. The charge distribution is also studied from the viewpoint of quantum multipion states, following Horn and Silver [1]. The modification of the distribution is attributed to the fact that for the general nonuniform solution, more than one orbital state is available for pions to occupy so that there are many different ways to construct multipion states.

2 Boost-Invariant Solution in $1+1$ Dimensions

In high energy hadron or nucleus collisions, the configurations approximately invariant along the collision axis are of particular interest. We first focus on this class of solutions ignoring the transverse spatial dependence. We choose a *nonlinear* σ-model as the dynamical model for QCD at low energy. The solutions that we obtain in this Section are equivalent to those of Blaizot and Krzywicki [4] though they are dervied in a slightly different way in order to clarify a relation between the uniformly oriented solutions and the nonuniformly oriented ones which plays an important role when we extend our argument to the more general case later.

The phase and radial representation of the nonlinear σ-model is,

$$\Sigma(x) = e^{i\boldsymbol{\tau}\cdot\mathbf{n}(x)\theta(x)}. \tag{1}$$

No matter what values the classical phase fields take, the state remains at the bottom of the potential valley because $|\Sigma| = 1$. This facilitates greatly the search for the DCC-type solutions which are realized at the bottom of potential well. Define the pion field

$$\boldsymbol{\pi}(x) = f_\pi \mathbf{n}(x)\,\theta(x). \tag{2}$$

where $\mathbf{n}(x)$ is an unit isovector field obeying $\mathbf{n}(x)\cdot\mathbf{n}(x) = 1$. Alternatively one defines $\boldsymbol{\pi}(x)$ by $\sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi} = f_\pi\Sigma$ with the constraint $\sigma = \sqrt{f_\pi^2 - \boldsymbol{\pi}^2}$. In this case the pion fields are given by $\boldsymbol{\pi}(x) = f_\pi\mathbf{n}(x)\,\sin\theta(x)$. In either case, $\mathbf{n}$ determines the isospin orientation of the pion field.

The lagrangian is given by

$$\mathcal{L} = \frac{f_\pi^2}{4}\mathrm{tr}\Big(\partial_\mu\Sigma^\dagger(x)\partial^\mu\Sigma(x)\Big), \tag{3}$$

where Σ transforms like $\Sigma \to U_L\Sigma U_R^\dagger$ under $SU(2)_L \times SU(2)_R$ rotations. In terms of $\theta(x)$ and $\mathbf{n}(x)$, the lagrangian is

$$\mathcal{L} = \frac{f_\pi^2}{2}(\partial_\mu\theta\,\partial^\mu\theta + \sin^2\theta\partial_\mu\mathbf{n}\cdot\partial^\mu\mathbf{n}) + \frac{\lambda f_\pi^2}{2}(\mathbf{n}^2 - 1), \tag{4}$$

where λ is a Lagrange multiplier. We will not include an explicit chiral symmetry breaking throughout this paper. The Euler-Lagrange equations are

$$\Box\theta = \sin\theta\,\cos\theta\,\partial_\mu\mathbf{n}\cdot\partial^\mu\mathbf{n}, \tag{5}$$

$$\partial_\mu(\sin^2\theta\,\partial^\mu \mathbf{n}) = \lambda \mathbf{n}. \tag{6}$$

The chiral $SU(2)_L \times SU(2)_R$ symmetry assures the conservation of the vector and axial-vector currents. In terms of θ and $\mathbf{n}$, the current conservation is written as

$$\partial_\mu(\sin^2\theta\,\mathbf{n}\times\partial^\mu \mathbf{n}) = 0, \tag{7}$$

$$\partial_\mu(\mathbf{n}\,\partial^\mu\theta + \sin\theta\,\cos\theta\,\partial^\mu \mathbf{n}) = 0. \tag{8}$$

The isospin current conservation (7) follows also from (6), while the axial-vector current conservation (8) can be derived from (5) by repeated use of $(\mathbf{n}\cdot d\mathbf{n}/d\tau) = 0$. (4), (5), (6) and (7) are most general with no assumptions or approximations made.

We consider a boost-invariant case in 1+1 dimensions where the fields $\theta(x)$ and $\mathbf{n}(x)$ are only functions of the variable τ:

$$\tau = \sqrt{t^2 - x^2}. \tag{9}$$

For a function only of τ, a partial derivative $\partial_\mu f(\tau)$ is equal to $(x_\mu/\tau)df/d\tau$. Furthermore, $\partial_\mu(f(\tau)\partial^\mu g(\tau)) = (1/\tau^2)(d(\tau^2 f g')/d\tau)$ where $g' = dg/d\tau$. The current conservation relations can be integrated into

$$\tau\,\sin^2\theta\,\mathbf{n}\times\frac{d\mathbf{n}}{d\tau} = \mathbf{a}, \tag{10}$$

$$\tau\,\mathbf{n}\frac{d\theta}{d\tau} + \tau\,\sin\theta\,\cos\theta\,\frac{d\mathbf{n}}{d\tau} = \mathbf{b}, \tag{11}$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors in the isospin space whose magnitudes are denoted as a and b respectively. It is immediately obvious from (10) and (11) that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal to each other:

$$\mathbf{a}\perp\mathbf{b}. \tag{12}$$

The isovector field $\mathbf{n}(\tau)$ stays perpendicular to $\mathbf{a}$ as τ varies. By multiplying (10) with $\mathbf{n}$ vectorially and using $\mathbf{n}\cdot(d\mathbf{n}/d\tau) = 0$, one obtains

$$\frac{d\mathbf{n}}{d\tau} = \frac{\mathbf{a}\times\mathbf{n}}{\tau\sin^2\theta}, \tag{13}$$

a standard equation for a vector $\mathbf{n}$ to precess around a constant vector $\mathbf{a}$. The precession frequency $|\mathbf{a}|/\tau \sin^2\theta$ varies with the proper time τ. The relations among $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{n}$ are illustrated in Figure 1. Squaring (11) gives

$$\left(\tau\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\,\cos^2\theta\left(\tau\frac{d\mathbf{n}}{d\tau}\right)^2 = b^2. \tag{14}$$

Eliminating $d\mathbf{n}/d\tau$ from the these equations, one obtains the differential equation for $\theta(\tau)$:

$$\left(\tau\frac{d\theta}{d\tau}\right)^2 = a^2 + b^2 - \frac{a^2}{\sin^2\theta}, \tag{15}$$

where $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$. (13) and (15) combined contain the same information as the first integrals of the Euler-Lagrange equations for θ and $\mathbf{n}$ so that we may proceed with the current conservation laws.

(15) is analytically integrable into the most general boost-invariant solution for $\theta(\tau)$ in 1+1 dimension:

$$\cos\theta(\tau) = (b/\kappa)\ \cos\Big(\kappa\ln(\tau/\tau_0) + \vartheta_0\Big), \tag{16}$$

where $\kappa^2 = a^2 + b^2$ and $\cos\vartheta_0 = (\kappa/b)\ \cos\theta(\tau_0)$. Substituting $\theta(\tau)$ in the integral from the isovector current conservation, one obtains the general solution for $\mathbf{n}(\tau)$. Since $\mathbf{n}(\tau)$ and $\mathbf{b}$ both lie in the plane perpendicular to $\mathbf{a}$, it is convenient to express the unit isovector field $\mathbf{n}(\tau)$ in terms of a single angle $\beta(\tau)$ measured from the direction of $\mathbf{b}$:

$$\mathbf{n}(x)\cdot\mathbf{b} = b\cos\beta(x). \tag{17}$$

The solution for $\cos\beta(\tau)$ can be expressed in terms of $\theta(\tau)$

$$\cos\,\beta(\tau) = \frac{a}{b}\sqrt{\frac{\kappa^2}{a^2} - \frac{1}{\sin^2\theta(\tau)}}. \tag{18}$$

The DCC configuration may be obtained from the boost-invariant solution multiplied by a step function $\Theta(\tau^2)$:

$$\Sigma(x)_{DCC} = e^{i\theta(x)\boldsymbol{\tau}\,\cdot\mathbf{n}(x)}\Theta(\tau^2), \tag{19}$$

such that the causality condition is satisfied. Our solutions should only apply to the inside of the light-cone. Once $\Theta(\tau^2)$ is inserted, there appears a source on the light cone at $\tau = 0$ which triggers the formation of a DCC. The energy density $\mathcal{E}(x)$ of our solution is singular as we approach the light cone:

$$\mathcal{E}(x) = \frac{f_\pi^2}{2}\left(\frac{t^2 + z^2}{\tau^2}\right)(a^2 + b^2), \tag{20}$$

for both the uniform and nonuniform solutions. The isospin vectors $\mathbf{a}$ and $\mathbf{b}$ enter the energy density in the combination of $a^2 + b^2$ as required by $SU(2) \times SU(2)$ invariance. The lowest energy solution of $a^2 + b^2 = 0$ is a trivial solution obtained from (16), (17) and (18) by taking the limit of $a, b \to 0$:

$$\theta(x) = \text{constant}, \ \mathbf{n}(x) = \text{constant vector}. \tag{21}$$

Blaizot and Krzywicki [4] expressed the pion fields by $\pi = f_\pi \mathbf{n} \sin\theta$. If we choose our *initial* condition such that $\sin\vartheta_0 = 0$, that is, $\cos\theta(\tau_0) = b/\kappa$, our general solution given by (16) and (18) coincides with theirs.

3 Chiral Rotation and Nonuniform Solution

In this Section we focus on the relation between the space-time dependence of the isovector field $\mathbf{n}$(x) and $SU(2) \times SU(2)$ invariance of the lagrangian. It is easy to see in (18) that our solutions have a uniform isospin orientation when the isospin vector $\mathbf{a}$ vanishes. When $\mathbf{a} = 0$, $\beta(\tau) = 0 \pmod{2\pi}$, that is, $\mathbf{n}$ points to the direction of $\mathbf{b}$ for all uniform DCC's. Because of the $SU(2) \times SU(2)$ invariance, it is always possible to rotate the vector $\mathbf{a}$ by an appropriate axial rotation to the direction of the vector $\mathbf{b}$. After the rotation the solution has a uniform isospin orientation and is degenerate in energy with the nonuniform solution prior to the rotation. It is not unfamiliar that if a system possesses some symmetry, a set of infinitely many new solutions may be obtained by making the symmetry transformation on a single solution.

To be explicit in the present case, let us rotate a uniform solution

$$\Sigma_0(x) = e^{i\theta_0(x)\boldsymbol{\tau}\cdot\mathbf{n}_0}, \tag{22}$$

where $\mathbf{n}_0$ is space-time independent. Upon a general chiral rotation parametrized by $U_L = e^{i\xi\boldsymbol{\tau}\cdot\mathbf{n}_L}$ and $U_R = e^{i\eta\boldsymbol{\tau}\cdot\mathbf{n}_R}$, the uniform solution is rotated into $\Sigma(x) = U_L\Sigma_0(x)U_R^{-1}$. The transformed θ and $\mathbf{n}$ fields are given by

$$\begin{aligned} c_\theta &= \Big(c_\xi c_\eta + (\mathbf{n}_L\cdot\mathbf{n}_R)s_\xi s_\eta\Big)c_0 + \\ &\quad \Big((\mathbf{n}_0\cdot\mathbf{n}_R)c_\xi s_\eta - (\mathbf{n}_0\times\mathbf{n}_L)s_\xi c_\eta + (\mathbf{n}_0\times\mathbf{n}_L)\cdot\mathbf{n}_R\, s_\xi s_\eta\Big)s_0, \end{aligned} \tag{23}$$

and

$$\begin{aligned} \mathbf{n}s_\theta &= \Big(\mathbf{n}_L s_\xi c_\eta - \mathbf{n}_R c_\xi s_\eta + (\mathbf{n}_L\times\mathbf{n}_R)s_\xi s_\eta\Big)c_0 \\ &\quad + \Bigg(\mathbf{n}_0\, c_\xi c_\eta + (\mathbf{n}_0\times\mathbf{n}_L)s_\xi c_\eta + (\mathbf{n}_0\times\mathbf{n}_R)c_\xi s_\eta \\ &\quad + \Big((\mathbf{n}_0\cdot\mathbf{n}_L)\mathbf{n}_R + (\mathbf{n}_0\cdot\mathbf{n}_R)\mathbf{n}_L - (\mathbf{n}_L\cdot\mathbf{n}_R)\mathbf{n}_0\Big)s_\xi s_\eta\Bigg)s_0, \end{aligned} \tag{24}$$

where c_θ and s_θ stand for $\cos\theta$ and $\sin\theta$, respectively, and so forth, while $c_0 = \cos\theta_0$ and $s_0 = \sin\theta_0$. For an isospin rotation, we choose $\xi = \eta$ and $\mathbf{n}_L = \mathbf{n}_R$. Then the rotated fields are

$$\begin{aligned} \cos\theta(\tau) &= \cos\theta_0(\tau), \\ \mathbf{n} &= (\mathbf{n}_0\cdot\mathbf{n}_L)\mathbf{n}_L - \Big(\mathbf{n}_0 - (\mathbf{n}_0\cdot\mathbf{n}_L)\mathbf{n}_L\Big)\cos 2\xi + (\mathbf{n}_0\times\mathbf{n}_L)\,\sin 2\xi. \end{aligned} \tag{25}$$

Since this is a global isospin rotation, the resulting field is another uniformly oriented solution with the same $\theta(x)$. For an axial rotation, $\xi = \eta$ and $\mathbf{n}_L = -\mathbf{n}_R$, in particular, if $\mathbf{n}_L, \mathbf{n}_R \perp \mathbf{n}_0$, one obtains

$$\begin{aligned} \cos\theta(\tau) &= \cos 2\xi\, \cos\theta_0(\tau), \\ \mathbf{n}(\tau)\,\sin\theta(\tau) &= \mathbf{n}_0\,\sin\theta_0(\tau) + \mathbf{n}_L\,\sin 2\xi\,\cos\theta_0(\tau). \end{aligned} \tag{26}$$

The axial rotations turn a uniform solution into nonuniform solutions. We can actually show that our general nonuniform solution given by (16) and (18) is reproduced with a suitable choice of the rotation angle:

$$\tan 2\xi = a/b. \tag{27}$$

Then (26) gives

$$\begin{aligned}\cos\theta(\tau) &= (b/\kappa)\cos\theta_0(\tau), \\ \cos\,\beta(\tau) &= \mathbf{n}\cdot\mathbf{n}_0 = \sin\theta_0(\tau)/\sin\theta(\tau).\end{aligned} \tag{28}$$

The result in (18) for $\cos\beta$ is obtained by solving the above equations. In this way we are able to obtain the nonuniform solutions from uniform ones by the axial-vector rotations. In the boost-invariant 1+1 dimensional case, we have obtained in Sec. 2 all nonuniform solutions by solving explicitly the nonlinear differential equations. We have shown that they are all related by the axial rotations to the nonuniform solutions with the same energy density $\mathcal{E}(x)$. All possible solutions are exhausted in this way.

4 General Solution Without Boost Invariance

Even without boost invariance, it is straightforward to solve for the uniformly oriented solutions. Once we obtain a uniform solution, we can transform it into the nonuniform solutions by $SU(2)\times SU(2)$ rotations. The equations for $\theta(x)$ and $\mathbf{n}(x)$ from (5) to (8) in Sec. 2 are also valid in the boost-noninvariant case. We look for the uniform solution in which $\mathbf{n}=$ constant so that $\partial_\mu\mathbf{n}=0$. In this case the equation of motion reduces simply to

$$\Box\theta = 0. \tag{29}$$

It is convenient to use the space-time variables

$$\tau=\sqrt{t^2-z^2}, \quad \eta=\frac{1}{2}\ln\left(\frac{t+z}{t-z}\right), \quad \mathbf{x}_\perp. \tag{30}$$

The origin of space-time coordinates is identified with the collision point of the hadron collisions. The z-axis is chosen along the collision axis of the initial hadrons. Note that the meaning of variable τ is a little different from the 1+1 dimensional case. The surface of $\tau=0$ lies outside the light cone with respect to the space-time origin except for the exactly forward and backward directions. With these space-time variables, $\Box\theta=0$ becomes

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial\theta}{\partial\tau}\right)-\frac{1}{\tau^2}\frac{\partial^2\theta}{\partial^2\eta}-\triangle_\perp\theta=0. \tag{31}$$

Since the differential equation is homogeneous, it can be solved by the method of separation of variables in the form

$$\theta(x) = T(\tau)H(\eta)X(\mathbf{x}_\perp)\,\theta(\tau^2 - \mathbf{x}_\perp^2). \tag{32}$$

We solve for $\theta(x)$ inside the light cone, $\tau^2 - \mathbf{x}_\perp^2 > 0$.

For the transverse direction $\mathbf{x}_\perp$, the general solutions are the Bessel and the Neumann functions. If we require that the solution be regular on the collision axis $\rho = |\mathbf{x}_\perp| = 0$, the Neumann functions are excluded. Note however that a singular behavior means an infinite oscillation toward $\rho = 0$, not an indefinite increase, in terms of the pion field $\boldsymbol{\pi} = f_\pi \mathbf{n}\sin\theta$. One may also require that $X(\mathbf{x}_\perp)$ should not increase indefinitely as $\rho \to \infty$. With these requirements, the parameter μ^2 defined by $\triangle_\perp X = -\mu^2 X$ must be positive. We choose $\mu > 0$. $X(\mathbf{x}_\perp)$ is expressed in terms of the Bessel functions of integer order:

$$X(\mathbf{x}_\perp) = C_0 J_0(\mu\rho) + \sum_{m=1}^{\infty} J_m(\mu\rho)(C_m \cos m\phi + D_m \sin m\phi), \tag{33}$$

where C_0, C_m, and D_m are the numerical coefficients to be determined by the boundary conditions. The magnitude of μ determines a transverse size of a DCC and therefore a spread of the p_t distribution of the final pions. Since a DCC will have an extended size in the transverse direction, the value of μ is likely to be a fraction of f_π or less.

The spatial rapidity dependence $H(\eta)$ is simply solved

$$H(\eta) = \cosh\lambda\eta \quad or \quad \sinh\lambda\eta. \tag{34}$$

The parameter λ can be any complex number in general (If it is complex, one should take the real part of $\theta(x)$ at the very end). The special case $\lambda = 0$ leads to the boost-invariant solutions. For the approximately boost-invariant DCC configurations, the magnitude of λ is much smaller than unity. The region of large values of η corresponds to the forward and backward edges of DCC where energetic leading hadrons are moving outward while the small values of η describe the cool central region. In this picture, it appears appropriate to choose $H(\eta)$ such that the pion density is higher at a larger η than at a smaller η. We therefore choose λ to be real (and positive) in the

following. It should be emphasized however that our choice for a real λ over purely imaginary or complex λ is more for the convenience of the presentation.

Given $X(\mathbf{x}_\perp)$ and $H(\eta)$, $T(\tau)$ obeys the differential equation

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial T}{\partial\tau}\right)+\left(\mu^2-\frac{\lambda^2}{\tau^2}\right)T=0. \tag{35}$$

The solution is given by $J_\lambda(\mu\tau)$ and/or $N_\lambda(\mu\tau)$. The main difference between $J_\lambda(\mu\tau)$ and $N_\lambda(\mu\tau)$ is their different behavior as $\mu\tau$ approaches 0. In the limit of a boost-invariance, $\lambda\to 0$, $J_\lambda(\mu\tau)$ approaches unity while $N_\lambda(\mu\tau)\to\ln(\mu\tau)$. If there is no transverse momentum, the latter approaches the uniform solution ($a=0$, $\kappa=b$) described in Sec. 2, while the former coincides with the lowest energy solution given in (21). Putting all together, one obtains the uniform solution in a complete form:

$$\begin{aligned}\theta(x)\mathbf{n}_0 \;=\;& \Big(a\;J_\lambda(\mu\tau)+b\;N_\lambda(\mu\tau)\Big)\\ &\times\Big(A\;\cosh\lambda\eta+B\;\sinh\lambda\eta\Big)\\ &\times J_m(\mu\rho)(C_m\cos m\phi+D_m\sin m\phi)\,\mathbf{n}_0\;\Theta(\tau^2-\mathbf{x}_\perp^2),\end{aligned} \tag{36}$$

where one may superpose these solutions in λ, μ, and m.

It is much harder to solve directly for nonuniform solutions when there is no boost-invariance constraint. Though some simple special solutions can be obtained by luck, finding all nonuniform solutions is a formidable task. In contrast, it is straightforward to perform $SU(2)\times SU(2)$ rotations on the uniform solutions. The rotation formulas (23) and (24) are most general and applicable to the boost-noninvariant case as well. Therefore the nonuniform solutions can be obtained by the chiral rotation from the uniform one in (36).

An important question is whether we exhaust all nonuniform solutions by the axial rotations from the uniform solutions. In other words, are there any nonuniform solutions that cannot be rotated into a uniform one? If such a class of solutions exists, it would have some topological quantum number like a soliton. Note however that the solutions of our interest are time-dependent, and that their energies and actions are not necessarily finite. Unless these topologically nontrivial solutions exist, the uniform solution (36) and the $SU(2)\times SU(2)$ rotations on it exhaust all solutions.

5 Pion Charge Distribution

It has been predicted that the pions decaying from a DCC will show a distinct charge distribution when the charge ratio is plotted event by event in $f = N_{\pi^0}/(N_{\pi^0} + N_{\pi^\pm})$. The distribution

$$\frac{dP}{df} = \frac{1}{2\sqrt{f}} \tag{37}$$

has been derived in two very different ways. The first derivation assumes that an isosinglet multipion state is created by the decay of a DCC [1]. All pions decaying from a given DCC are assumed to occupy an identical orbital state that is determined by the spatial configuration of DCC. The Bose statistics allows only one isosinglet $2N$-pion state:

$$|2N\pi\rangle = (2a_+^\dagger a_-^\dagger - a_0^\dagger a_0^\dagger)^N|0\rangle, \tag{38}$$

where $a_{\pm,0}^\dagger$ are the creation operators of the pions in the same single orbital state. Making a binomial expansion of the right-hand side at large N, one obtains a simple rule $dP/df = 1/(2\sqrt{f})$. It is later pointed out that the relative phase between $a_+^\dagger a_-^\dagger$ and $a_0^\dagger a_0^\dagger$ is inessential to the final prediction of dP/df [6].

The second derivation is based on a more intuitive picture in classical field theory. Assuming that the isospin orientation is uniform in space-time and that all isospin directions are equally probable, one obtains $dP/d\Omega = 1/4\pi$, where Ω is the solid angle for an isospin direction in isospin space. Since the π^0 fraction f is proportional to the square of the third component of the pion field, ($f \propto \cos^2\beta$), one obtains again distribution (37). In this derivation, the interference effects are completely ignored.

Let us examine whether or not this prediction remains valid for the nonuniform DCC's. In the first derivation, it is crucial that only one orbital state is available for pions and therefore the isosinglet state is unique: For two pions, the isosinglet is nothing but $(2a_+^\dagger a_-^\dagger - a_0^\dagger a_0^\dagger)|0\rangle$ by the Clebsch-Gordan coefficients. For four pions, the group theory alone would allow two isosinglets. One is to combine the $\mathbf{0}_{2\pi}$ from $\mathbf{1}_\pi \otimes \mathbf{1}_\pi = \mathbf{0}_{2\pi} + \mathbf{1}_{2\pi}$ with the other $\mathbf{0}_{2\pi}$ from $\mathbf{1}_\pi \otimes \mathbf{1}_\pi = \mathbf{0}_{2\pi} + \mathbf{1}_{2\pi}$. The other is to contract the $\mathbf{1}_{2\pi}$ from $\mathbf{1}_\pi \otimes \mathbf{1}_\pi = \mathbf{0}_{2\pi} + \mathbf{1}_{2\pi}$ with the other $\mathbf{1}_{2\pi}$ from $\mathbf{1}_\pi \otimes \mathbf{1}_\pi = \mathbf{0}_{2\pi} + \mathbf{1}_{2\pi}$. The Bose statistics forbids $\mathbf{1}_{2\pi}$ for two identical pions in the same orbital state so that only $\mathbf{0}_{2\pi} \otimes \mathbf{0}_{2\pi}|0\rangle = (2a_+^\dagger a_-^\dagger - a_0^\dagger a_0^\dagger)^2|0\rangle$ is allowed. This argument goes through for any

$2N$, leading to the $|2N\pi\rangle$ in (38). If there are more than one orbital states available, the four-pion singlet state would generally take the form

$$|4\pi\rangle = \Big(A(\mathbf{0}_{2\pi} \otimes \mathbf{0}_{2\pi}) + B(\mathbf{1}_{2\pi} \otimes \mathbf{1}_{2\pi})\Big)|0\rangle, \tag{39}$$

where the coefficients A and B are dynamics-dependent. There are increasingly many more isosinglets for 6π's, 8π's *etc*, as N goes up. In the above example, the A-type term and the B-type term give quite different pion compositions: there is $\pi^0\pi^0\pi^0\pi^0$ in the A-type term, but no $\pi^0\pi^0\pi^0\pi^0$ in the B-type term. In order to obtain the distribution $dP/df = 1/(2\sqrt{f})$, there must be only the A-type term and nothing else in the $2N\pi$ state ($N \to \infty$).

One can construct explicitly the 4π state when the isovector field $\mathbf{n}(x)$ is nonuniform in space-time. Let us parametrize the direction of $\mathbf{n}(x)$ by the azimuthal and polar angles $\alpha(x)$ and $\beta(x)$ with respect to the isospin z-axis. To simplify our computation a little, we consider as an example a DCC whose isospin is nonuniform only in the polar direction β, but not in the azimuthal direction by choosing $\alpha = 0$. We shall use the representation $\boldsymbol{\pi}(x) = f_\pi \mathbf{n}(x)\sin\theta(x)$ instead of $\pi(x) = f_\pi \mathbf{n}(x)\theta(x)$ for the following discussion since the former automatically incorporates the periodicity of $\Sigma(x)$ in $\theta(x) \to \theta(x) \pm 2\pi$. The Cartesian isospin components of the pion field are

$$\begin{aligned}
\pi_1 &= f_\pi \sin\theta(x)\ \sin\beta(x), \\
\pi_2 &= 0, \\
\pi_3 &= f_\pi \sin\theta(x)\ \cos\beta(x).
\end{aligned} \tag{40}$$

The DCC state is described by the quantum coherent state, up to an overall normalization

$$|DCC(\theta,\beta)\rangle = \exp\Big(a_1^\dagger(s_\theta s_\beta) + a_3^\dagger(s_\theta c_\beta)\Big)|0\rangle, \tag{41}$$

where

$$\begin{aligned}
a_1^\dagger(s_\theta s_\beta) &= \int \sqrt{2|\mathbf{k}|}\phi_{ss}(\mathbf{k}) a_1^\dagger(\mathbf{k}) d^3\mathbf{k}, \\
a_3^\dagger(s_\theta c_\beta) &= \int \sqrt{2|\mathbf{k}|}\phi_{sc}(\mathbf{k}) a_3^\dagger(\mathbf{k}) d^3\mathbf{k},
\end{aligned} \tag{42}$$

with ϕ_{ss} and ϕ_{sc} being the three-dimensional Fourier transforms of $f_\pi \sin\theta \sin\beta$ and $f_\pi \sin\theta \cos\beta$ respectively. Unlike $a_i^\dagger(\mathbf{k})$, the operators $a_1^\dagger(s_\theta s_\beta)$ and $a_3^\dagger(s_\theta c_\beta)$ are not canonically normalized, but the normalization is irrelevant to the isospin structure. The $|N\pi\rangle$ projection of the DCC state is

$$|N\pi(\theta(x)\,\beta(x))\rangle = \frac{1}{N!}\Big(a_1^\dagger(s_\theta\, s_\beta) + a_3^\dagger(s_\theta\, c_\beta)\Big)^N |0\rangle. \tag{43}$$

Under the assumption that the DCC's appear in the intermediate state with $I = 0$ and the production processes conserve isospin, if one DCC can be produced, all other DCC's that are related to it by the isospin rotations can be produced with an equal probability. The isosinglet DCC state can be constructed from the state in (41) by integrating out the Euler angles over the entire isospin space.

The 4π state of an isosinglet DCC is obtained by averaging $|4\pi(\theta(x)\,\beta(x))\rangle$ over isospin space. The computation is straightforward though a little tedious. Up to an overall normalization, the result is

$$|4\pi(I=0)\rangle = \left(\left(\Big|\mathbf{a}^\dagger(s_\theta c_\beta)\Big|^2 + \Big|\mathbf{a}^\dagger(s_\theta s_\beta)\Big|^2\right)^2 - 4\Big|\mathbf{a}^\dagger(s_\theta c_\beta)\times\mathbf{a}^\dagger(s_\theta s_\beta)\Big|^2\right)|0\rangle, \tag{44}$$

where $|\mathbf{a}^\dagger|^2 = 2a_+^\dagger a_-^\dagger - a_0^\dagger a_0^\dagger$. The first and second terms in the right-hand side are the A-type terms in (39), while the last term is the B-type term. As we anticipate, the isosinglet 4π state of the nonuniform DCC is no longer of the form postulated in (38). For a uniform DCC, that is, $\beta(x) \to$ constant, $\mathbf{a}^\dagger(s_\theta c_\beta)$ and $\mathbf{a}^\dagger(s_\theta s_\beta)$ are identical up to a factor (in the 1+1 boost-invariant solution in Sec. 2, $\beta(x)$ is so defined that $\beta(x) \to 0$, namely $\mathbf{a}^\dagger \to 0$, in the uniform limit). Therefore, the B-type term cannot exist for the uniform DCC's. Our construction of the isosinglet 4π state and the existence of the B-type terms cast a serious doubt on the distribution for the nonuniform DCC's.

Alternatively, let us study the problem by assuming that $|\mathrm{DCC}(\theta(x), \beta(x))\rangle$ with different $\theta(x)$ and $\beta(x)$ do not have the quantum interference with each other. It is in accordance with the classical field picture. For a large number of pions, ignoring the interference may be justified. The momentum spectrum of pion quanta ($i = 1, 2, 3$)

decaying from a classical field is given [7]

$$(2\pi)^3 \frac{dN_i}{d^3\mathbf{k}} = \frac{|\tilde{\rho}_i(\mathbf{k}, |\mathbf{k}|)|^2}{2|\mathbf{k}|}, \tag{45}$$

where $\tilde{\rho}_i(\mathbf{k}, |\mathbf{k}|)$ is the four-dimensional on-mass-shell Fourier transform of the pion source function $\rho_i(\mathbf{x}, t)$ defined by $\Box\boldsymbol{\pi}(x) = \boldsymbol{\rho}(x)$:

$$\tilde{\rho}_i(\mathbf{k}, |\mathbf{k}|) = \int \rho_i(\mathbf{x}, t)\, e^{-i\mathbf{k}\cdot\mathbf{x}+i|\mathbf{k}|t} d\mathbf{x}dt. \tag{46}$$

It is convenient to perform the space-time integral using variables τ, η and $\mathbf{x}_\perp$ for which $d\mathbf{x}dt = \tau d\tau d\eta d\mathbf{x}_\perp$, and

$$E = |\mathbf{k}|, \quad y = \frac{1}{2}\ln\left(\frac{E + k_\parallel}{E - k_\parallel}\right), \quad \mathbf{k}_\perp, \tag{47}$$

for the momentum variables. (45) becomes

$$(2\pi)^3 \frac{dN_i}{dy\, d^2\mathbf{k}_\perp} = \Big| \int \rho_i(\tau,\, \eta,\, \mathbf{x}_\perp)\; e^{i|\mathbf{k}_\perp|\tau\, \cosh(\eta-y) - i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} \tau\, d\tau\, d\eta\, d^2\mathbf{x}_\perp \Big|^2. \tag{48}$$

If a DCC is boost-invariant along the collision axis, $\rho(\tau, \eta, \mathbf{x}_\perp)$ does not depend on η. In this case, η is integrated out and the energy spectrum $dN_i/dy\, d^2\mathbf{k}_\perp$ is independent of the rapidity variable y, as it is well known.

Let us look into the isospin structure of the Fourier transform of the source that enters the right-hand side of (48). After the space-time integration is performed, the isovector $\tilde{\boldsymbol{\rho}}$ is generally of the form

$$\tilde{\boldsymbol{\rho}}(\mathbf{k}, |\mathbf{k}|) = F(y, \mathbf{k}_\perp)\boldsymbol{e}(\mathbf{k})\ , \tag{49}$$

where $F(y, \mathbf{k}_\perp)$ is an isoscalar, Lorentz-scalar function of $\mathbf{k}$ and of whatever parameters that characterize a DCC; $\boldsymbol{e}(\mathbf{k})$ is a unit vector in isospin space. The pion spectrum is simply

$$(2\pi)^3 \frac{dN_i}{dy\, d^2\mathbf{k}_\perp} = |F(y, \mathbf{k}_\perp)e_i(\mathbf{k})|^2. \tag{50}$$

For each *fixed* $\mathbf{k}$, one may repeat the classical field derivation and reproduce

$$\frac{dP}{df(\mathbf{k})} = \frac{1}{2\sqrt{f(\mathbf{k})}}. \tag{51}$$

However, it is clear that the pion number N_i no longer obeys distribution (37) when the momentum $\mathbf{k}$ is integrated over. To illustrate this point, consider a toy DCC for which $\boldsymbol{e}(\mathbf{k})$ points to one direction for a half of the range of rapidity y and to another direction perpendicular to it for the other half of y. Such a DCC is not one of the solutions that we have obtained, but it serves to make a point. Since there is no way to align the two $\mathbf{e}(\mathbf{k})$'s to the same direction by isospin rotation, there are no DCC's in this isospin family that emit only π^0, even though all directions are equally probable in isospin space. For a family of nonuniform DCC's, dP/df is zero at $f = 0$ (*Centauro*) and at $f = 1$(*anti-Centauro*), and tends to bulge in the central region of f, unlike that for a family of uniform DCC's. Only if the uniform DCC's dominate over the nonuniform ones, can distribution (37) hold approximately. The abundance of the uniform DCC's has a measure zero relative to that of the nonuniform DCC's in the phase space of the rotation angles. Unless the production of the nonuniform DCC's by the initial hadrons is strongly suppressed for some dynamical reason, $dP/df = 1/(2\sqrt{f})$ cannot hold even approximately. The spectacular Centauro and anti-Centauro events will be far rarer than our naive expectation based on the uniform DCC's. However, there may be a chance to observe distribution (51) by selecting pions of the same y and $\mathbf{k}_\perp$ within small uncertainties.

A special case is a boost-invariant DCC in 1 + 1 dimensions where $\boldsymbol{\rho}(x)$ is η-independent so that $\tilde{\boldsymbol{\rho}}(\mathbf{k})$ is y-independent. In this case, $\boldsymbol{e}(\mathbf{k})$ becomes a constant vector independent of $\mathbf{k}$ and distribution (37) follows even for the nonuniform DCC's. One the other hand, for the 4π state that we studied in this Section, we see nothing special about the boost-invariant nonuniform case with $\mathbf{k}_\perp = 0$ from the general boost-noninvariant case. Do two derivations contradict with each other? It is difficult to make a connection between the two arguments. In analyzing the $|N\pi(I = 0)\rangle$ state, the interference between different DCC's is essential while in the classical field analysis, each DCC state is not an eigenstate of isospin and the interference from different DCC's is completely discarded. Though the both methods have led to the same dP/df distribution, it is not clear how much similar or mutually compatible their physical pictures are. With this unsolved uncertainty, we state our conclusion in a less assertive way: if we follow the isospin analysis of $|2N\pi\rangle$, we see no mechanism

that leads to distribution (37) for the nonuniform DCC's. If we argue instead in the classical field picture, the distribution does not hold except for the boost-invariant DCC with zero $\mathbf{k}_\perp$. However, distribution (37) should hold for pions which are selected from a small segment of rapidity y and transverse momentum $\mathbf{k}_\perp$.

Acknowledgment

We wish to thank J.D. Bjorken for giving us his notes on his solutions and many useful discussions. This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under contract DE-AC03-76SF00098 and in part by the U.S. National Science Foundation under grant PHY-90-21139. One of us (Z.H.) acknowledges the support from the Natural Sciences and Engineering Research Council of Canada.

References

[1] D. Horn and R. Silver, Ann. Phys.(N.Y.) **66**, 509 (1971)

[2] A. A. Anselm and M. G. Ryskin, Phys. Lett. B **266**, 482 (1991); J. -P. Blaizot and A. Krzywcki, Phys. Rev. D **46**, 246 (1992); J. D. Bjorken, K. L. Kowalski, and C. C. Taylor, SLAC preprint SLAC-PUB-6109.

[3] K. Rajagopal and F. Wilczek, Nucl. Phys. **B399**, 395 (1992); *ibid* **B404**, 577 (1993).

[4] J.-P. Blaizot and A. Krzywicki, Phys. Rev. **D46**, 246 (1992)

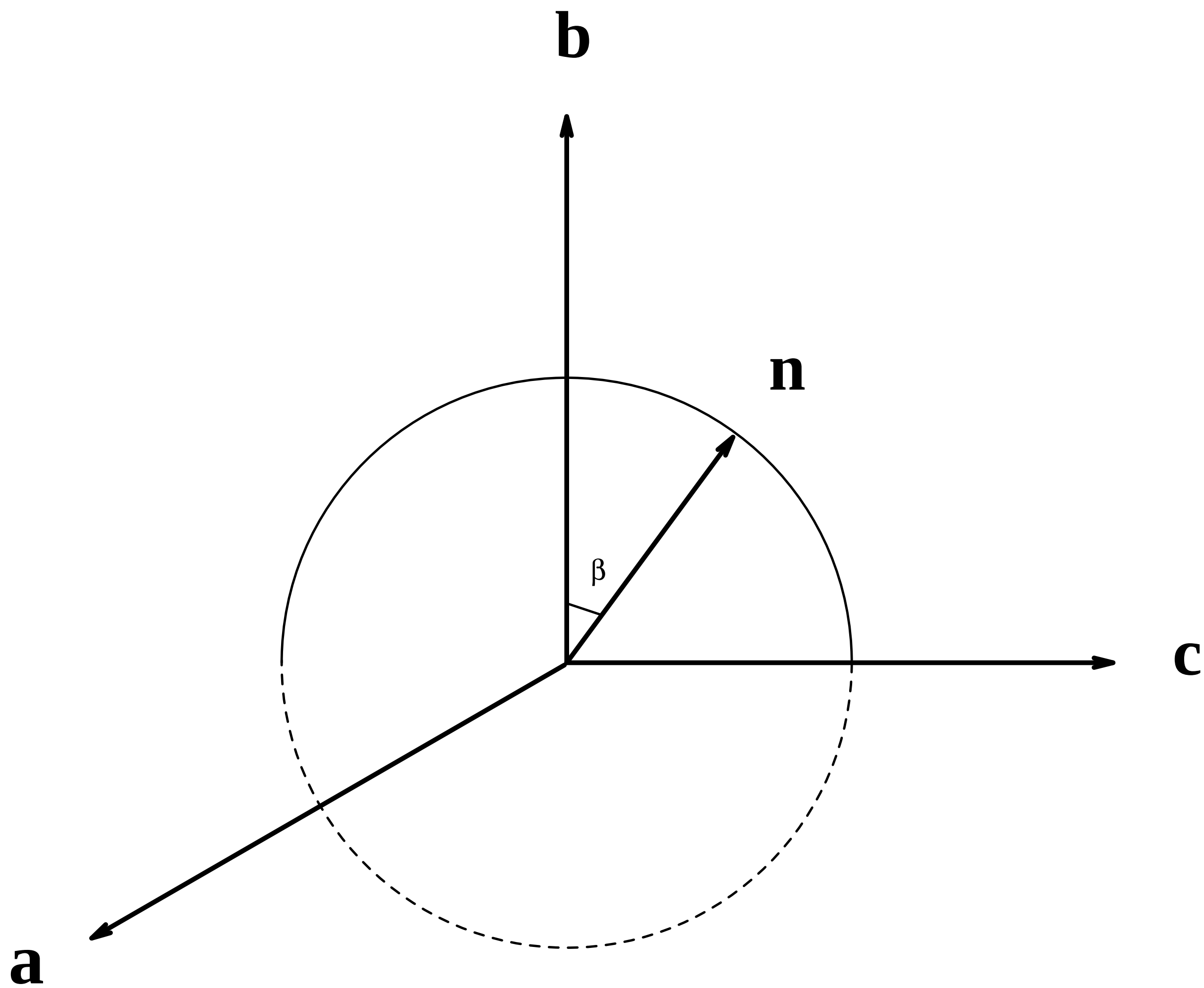
[5] A. A. Anselm and M. Bander, Pis'ma Zh. Eksp. Teor. Fiz. **59**, 479 (1994) [JETP Lett. **59**, 503 (1994)].

[6] I.I. Kogan, Pis'ma Zh. Eksp. Teor. Fiz. **59**, 289 (1994) [JETP Lett. **59**, 307 (1994)]

[7] E.M. Henley and W. Thirring, *Elementary Quantum Field Theory* (McGraw Hill, New York, 1962), Chapter 8-10

Figure caption

Figure 1: The direction of $\mathbf{n}(\tau)$ relative to $\mathbf{a}$ and $\mathbf{b}$. As τ varies, $\mathbf{n}(\tau)$ precesses in the plane perpendicular to the vector $\mathbf{a}$. $\mathbf{c} = \mathbf{b} \times \mathbf{a}$. For the uniform solutions, $\mathbf{n}$ stays in the direction of $\mathbf{b}$ for all τ.


b
n
β
c
a